Distinguishing two-qubit states using local measurements and restricted classical communication

Mark Hillery and Jihane Mimih
Department of Physics and Astronomy
Hunter College of CUNY
695 Park Avenue
New York, NY 10021

October 29, 2002

Abstract

The problem of unambiguous state discrimination consists of determining which of a set of known quantum states a particular system is in. One is allowed to fail, but not to make a mistake. The optimal procedure is the one with the lowest failure probability. This procedure has been extended to bipartite states where the two parties, Alice and Bob, are allowed to manipulate their particles locally and communicate classically in order to determine which of two possible two-particle states they have been given. The failure probability of this local procedure is the same as if the two particles were together in the same location. Here we examine the effect of restricting the classical communication between the parties, either allowing none or eliminating the possibility that one party's measurement depends on the result of the other party's. These issues are studied for two-qubit states, and optimal procedures are found. In some cases the restrictions cause increases in the failure probability, but in other cases they do not. Applications of these procedure, in particular to secret sharing, are discussed.

1 Introduction

Suppose that we have a two-qubit state, and we give one of the qubits to Alice and the other to Bob. Alice and Bob know that the state is either $|\Psi_0\rangle$ or $|\Psi_1\rangle$, and by making local measurements and communicating classically, they want to determine which state they have. We want to consider the case of unambiguous discrimination, which means that Alice and Bob may fail to decide which state they have, but if they succeed, they will not make an error. That is, they will never conclude that they have $|\Psi_0\rangle$ when they have been given $|\Psi_1\rangle$ and vice

versa. Our object is to develop a procedure that Alice and Bob can use to discriminate between the states.

One aspect of this problem has already been solved. If each state is equally likely and and both qubits can be measured together, then it is known that the states can be successfully unambiguously discriminated with a probability of $p_{idp} = 1 - |\langle\Psi_0|\Psi_1\rangle|$ [1]-[3]. It was recently shown that the states can be discriminated using only local operations and classical communication (LOCC) with the same success probability. Walgate, et al. proved that if $\langle\Psi_0|\Psi_1\rangle = 0$, then the states can be distinguished perfectly using only LOCC [4]. The case when $|\Psi_0\rangle$ and $|\Psi_1\rangle$ are not orthogonal was investigated numerically by Virmani, et al. [5], and they found strong evidence that unambiguous discrimination is possible with a probability of p_{idp} using LOCC. In addition, they found a class of states for which they could prove that this was true. A proof that this is true for all bipartite states was provided by Chen and Yang [6].

The procedure that makes LOCC unambiguous discrimination with a success probability of p_{idp} possible is the following. Alice makes a projective measurement on her particle that gives her no information about whether the state is $|\Psi_0\rangle$ or $|\Psi_1\rangle$, and she then communicates her result to Bob. Based on what Alice has told him, Bob chooses a measurement to make on his particle. In particular, he applies the procedure for the optimal unambiguous discrimination of single qubit states to his particle. However, in this procedure one must know the two states that one is discriminating between, and it is this information that is provided by the result of Alice's measurement.

What we wish to examine here is how restricting the classical communication between the parties affects their ability to discriminate between the states. We shall first see what happens when no classical communication is allowed. In that case each party has three possible measurement results, 0 corresponding to $|\Psi_0\rangle$, 1 corresponding to $|\Psi_1\rangle$, and f for failure to distinguish. If $|\Psi_0\rangle$ is sent, then Alice and Bob both measure 0 or both measure f, so that they both know, without communicating, that $|\Psi_0\rangle$ was sent or that the measurement failed. If $|\Psi_1\rangle$ is sent, then they both measure either 1 or f. We shall then relax the ban on classical commuincation, and allow Alice and Bob to communicate their meaurement results to each other. However, conditional measurements will still be banned, i.e. situations in which the measurement made by one party depends on the measurement results of the other will not be allowed.

One motivation for studying these situations, in addition to what they tell us about state discrimination, is their possible use in communication schemes. State discrimination for single qubits can be used to construct a scheme for quantum cryptography [7]. In this protocol, Alice and Bob wish to share a secure key. Alice sends single qubits to Bob in one of two nonorthogonal states, $|\psi_0\rangle$ or $|\psi_1\rangle$, and Bob applies the unambiguous state discrimination protocol to the states he receives. He then tells Alice whether the procedure succeeded or failed, and they keep the instances when it succeeded and throw out the rest. If Bob's measurement resulted in $|\psi_0\rangle$, then that particular key bit is recorded as 0, and if it resulted in $|\psi_1\rangle$, it is recorded as 1. In this way a binary string shared by Alice and Bob can be constructed, and it serves as the key.

An eavesdropper, Eve, who intercepts the qubits that Alice sends to Bob, and who wishes to find out which state they are in, has a problem. Because the states are not orthogonal, she will not be able to definitely determine the state each of each qubit she receives. However, she must send a qubit in either $|\psi_0\rangle$ or $|\psi_1\rangle$ on to Bob. Since her information about the qubit she received is not perfect, Eve will sometimes send a qubit in the wrong state to Bob. If Alice and Bob publicly compare some of their key bits and find discrepencies, then they know an eavesdropper was present. If they find no discrepencies, then they can conclude that they share a secure key.

The no-classical-communication scheme would allow a third party, Charlie, to distribute a shared key to Alice and Bob. Charlie would send one qubit to Alice and one to Bob, where the qubits are either in the state $|\Psi_0\rangle$ or $|\Psi_1\rangle$, and Alice and Bob would measure them. They would both know when the had found $|\Psi_0\rangle$, when they had found $|\Psi_1\rangle$, and when they had failed. Note that Charlie would not know the key, because he would not know which bits corresponded to failure. A slight relaxation of the no-classical-communication condition allows all three parties to share a key. Alice and Bob simply announce publicly when they failed to distinguish the state.

A possible use for the second scheme, when Alice and Bob are allowed to compare their meaurement results, is secret sharing. In this case a third party, Charlie, wants to share a secure key with Alice and Bob, but he wants Alice and Bob to have to cooperate to determine the key bit. Neither Alice nor Bob, separately, will know the key, but together they will. Charlie accomplishes this by sending one qubit to Alice and another to Bob. The two qubits are either in the state $|\Psi_0\rangle$ or $|\Psi_1\rangle$, and these states are not orthogonal. Alice and Bob then perform a procedure to determine which state they have, and this procedure must require their cooperation, so that neither of them by themselves can determine the state. If they use the optimal procedure in which the measurement Bob makes depends on the result of Alice's measurement, then Alice would measure her particle, and Bob would do nothing to his. When they want to determine the key bit, Alice will tell Bob the result of her measurement, and Bob will make the appropriate measurement on his particle. This method, however, requires Bob to store quantum information, i.e. keep his particle free from the effects of decoherence, until the bit is determined. A more practical procedure would be the restricted-classical-communication scheme in which both Alice and Bob make independent measurements, and are able to determine the state from the results. In that case, they each measure their qubit when they receive it, and they record the results of their measurements. This means that it is only classical information that needs to be stored. Neither Alice nor Bob should be able to determine the state from just their own result, but by putting their results together they should be able to indentify the state they were sent with some nonzero probability, and they should never make an error. It is this kind of procedure we wish to study here.

2 No classical communication

As discussed in the Introduction, we shall assume that Alice and Bob each has one of three measurement alternatives, 0, 1, and f. The POVM operators that characterize the measurements are $\{A_0, A_1, A_f\}$ for Alice and $\{B_0, B_1, B_f\}$ for Bob. These operators satisfy

$$I_A = \sum_{j=0,1,f} A_j^\dagger A_j \qquad I_B = \sum_{j=0,1,f} B_j^\dagger B_j, \tag{1}$$

where I_A is the identity on $\mathcal{H}_A$, the Hilbert space of Alice's qubit, and I_B is the identity on $\mathcal{H}_B$, the space of Bob's qubit. The requirement that Alice and Bob only get the same result for their measurements imposes the conditions

$$A_j B_k |\Psi_n\rangle = 0, \tag{2}$$

where $j, k \in \{0, 1, f\}$ and $j \neq k$, and $n \in \{0, 1\}$. In addition, the fact that no errors are made in identitfying the states requires that

$$A_0 B_0 |\Psi_1\rangle \qquad A_1 B_1 |\Psi_0\rangle = 0. \tag{3}$$

It is clear simply from the number of conditions, that if this procedure is possible at all, it will be true only for a very restricted set of states. In fact, what we find is that the best we can do is to is to identify one of the states with a nonzero probability and fail the rest of the time. The details of the proof of this statement are given in the Appendix.

We conclude this section with an example of the situation in which one state can be detected. Suppose our two states are given by

$$\begin{aligned} |\Psi_0\rangle &= |0\rangle|0\rangle \\ |\Psi_1\rangle &= \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle), \end{aligned} \tag{4}$$

where Alice's states are first and Bob's second. In addition, we have that $A_0 = B_0 = 0$, so that $|\Psi_0\rangle$ is never detected, and

$$\begin{aligned} A_1 &= |1\rangle\langle 1| \quad B_1 = |1\rangle\langle 1| \\ A_f &= |0\rangle\langle 0| \quad B_f = |0\rangle\langle 0|. \end{aligned} \tag{5}$$

From this we see that, indeed, if $|\Psi_0\rangle$ is sent, then it will not be detected, but if $|\Psi_1\rangle$ is sent, then we will detect it with a probability of $1/2$ and fail with a probability of $1/2$. Thus, we are very limited in distinguishing two states without any classical communication between Alice and Bob.

3 Limited classical communication

The situation becomes more interesting if we allow Alice and Bob to communicate the results of their measurements to each other only after both measurements have been made. We now consider the following situation. Alice and Bob

make measurements on their particles, and each of these measurements can have one of two outcomes, 0 or 1. Alice's measurement is described by the POVM $\{A_0, A_1\}$ and Bob's by $\{B_0, B_1\}$, where

$$I_A = A_0^\dagger A_0 + A_1^\dagger A_1 \qquad I_B = B_0^\dagger B_0 + B_1^\dagger B_1, \tag{6}$$

and I_A and I_B are the identity operators in Alice's and Bob's Hilbert spaces, respectively. The probability that Alice will obtain the result k if the two qubit-state is $|\Psi_j\rangle$ is $\mathrm{Tr}(\rho_{Aj} A_k^\dagger A_k)$, where $\rho_{Aj} = \mathrm{Tr}_B(|\Psi_j\rangle\langle\Psi_j|)$ is the reduced density matrix of $|\Psi_j\rangle$ in Alice's space. Similar expressions hold for the probabilities of Bob's measurements. Note that A_0 and A_1 commute with B_0 and B_1.

Together, Alice and Bob have four possible sets of results (Alice's result is given first, Bob's second), $\{0,0\}$, $\{0,1\}$, $\{1,0\}$, $\{1,1\}$, and we have to decide which sets correspond to $|\Psi_0\rangle$, which to $|\Psi_1\rangle$, and which to failure to decide. Let us first consider what happens if we assume that none of the sets corresponds to failure. In particular, suppose that $\{0,0\}$ and $\{1,1\}$ correspond to $|\Psi_0\rangle$ and $\{0,1\}$ and $\{1,0\}$ correspond to $|\Psi_1\rangle$. This implies that if the state is $|\Psi_1\rangle$, then the probability of getting $\{0,0\}$ or $\{1,1\}$ is zero, and if the state is $|\Psi_0\rangle$, the probability of getting $\{0,1\}$ or $\{1,0\}$ is zero. Therefore, we have

$$\begin{aligned} \langle\Psi_0|A_0^\dagger A_0 B_1^\dagger B_1|\Psi_0\rangle &= \langle\Psi_0|A_1^\dagger A_1 B_0^\dagger B_0|\Psi_0\rangle &= 0 \\ \langle\Psi_1|A_0^\dagger A_0 B_0^\dagger B_0|\Psi_1\rangle &= \langle\Psi_1|A_1^\dagger A_1 B_1^\dagger B_1|\Psi_1\rangle &= 0. \end{aligned} \tag{7}$$

These imply the simpler equations

$$\begin{aligned} A_0 B_1|\Psi_0\rangle &= A_1 B_0|\Psi_0\rangle &= 0 \\ A_0 B_0|\Psi_1\rangle &= A_1 B_1|\Psi_1\rangle &= 0. \end{aligned} \tag{8}$$

If we now note that

$$\begin{aligned} \langle\Psi_1|\Psi_0\rangle &= \langle\Psi_1|I_A \otimes I_B|\Psi_0\rangle \\ &= \langle\Psi_1|(A_0^\dagger A_0 + A_1^\dagger A_1) \otimes (B_0^\dagger B_0 + B_1^\dagger B_1)\Psi_0\rangle, \end{aligned} \tag{9}$$

we see from the previous equation that $\langle\Psi_1|\Psi_0\rangle = 0$. Therefore, if we are able to distinguish the states every time without error, they must be orthogonal.

Now let us suppose that some of the measurement results correspond to a failure to distinguish the states. We will focus on two different cases. In the first we shall assume that two of the four alternatives correspond to failure, and in the second we shall assume that only one does.

3.1 Two failure states

Let us assume that $\{0,0\}$ corresponds to $|\Psi_0\rangle$, $\{1,1\}$ corresponds to $|\Psi_1\rangle$, and both $\{0,1\}$ and $\{1,0\}$ correspond to failure to distinguish. The condition of no errors implies that

$$A_0 B_0|\Psi_1\rangle = 0 \qquad A_1 B_1|\Psi_1\rangle = 0. \tag{10}$$

If we apply these conditions to Eq. (9), we find that

$$\langle\Psi_1|\Psi_0\rangle = \langle\Psi_1|F|\Psi_0\rangle, \tag{11}$$

where

$$F = A_0^\dagger A_0 B_1^\dagger B_1 + A_1^\dagger A_1 B_0^\dagger B_0. \tag{12}$$

Now let us examine the conditions in Eq. (10) in more detail. We first express $|\Psi_1\rangle$ in its Schmidt basis

$$|\Psi_1\rangle = \sum_{j=0}^{1} \sqrt{\lambda_{1j}}|v_{Aj}\rangle \otimes |v_{Bj}\rangle, \tag{13}$$

where $\{v_{A0}, v_{A1}\}$ and $\{v_{B0}, v_{B1}\}$ are orthonormal bases for Alice's and Bob's spaces, respectively, and λ_{1j} for $j = 0, 1$ are the eigenvalues of the reduced density matrixes. The condition $A_0 B_0|\Psi_1\rangle = 0$ then implies that

$$\sqrt{\lambda_{10}}A_0|v_{A0}\rangle \otimes B_0|v_{B0}\rangle = -\sqrt{\lambda_{11}}A_0|v_{A1}\rangle \otimes B_0|v_{B1}\rangle. \tag{14}$$

The only way this can be true is if $A_0|v_{A0}\rangle$ is parallel to $A_0|v_{A1}\rangle$ and $B_0|v_{B0}\rangle$ is parallel to $B_0|v_{B1}\rangle$. Therefore, we can write

$$\begin{aligned} A_0|v_{A0}\rangle &= c_0|\eta_A\rangle \quad B_0|v_{B0}\rangle = d_0|\eta_B\rangle \\ A_0|v_{A1}\rangle &= c_1|\eta_A\rangle \quad B_0|v_{B1}\rangle = d_1|\eta_B\rangle, \end{aligned} \tag{15}$$

where c_j and d_j are constants and $\|\eta_A\| = \|\eta_B\| = 1$. These equation imply that

$$\begin{aligned} A_0 &= \sum_{j=0}^{1} c_j|\eta_A\rangle\langle v_{Aj}| = |\eta_A\rangle\langle r_A| \\ B_0 &= \sum_{j=0}^{1} d_j|\eta_B\rangle\langle v_{Bj}| = |\eta_B\rangle\langle r_B|, \end{aligned} \tag{16}$$

where

$$|r_A\rangle = \sum_{j=0}^{1} c_j^*|v_{Aj}\rangle \qquad |r_B\rangle = \sum_{j=0}^{1} d_j^*|v_{Bj}\rangle. \tag{17}$$

The condition $A_0 B_0|\Psi_1\rangle = 0$ can now be expressed as

$$(\langle r_A| \otimes \langle r_B|)|\Psi_1\rangle = 0. \tag{18}$$

We can now do the same thing with the condition that $A_1 B_1|\Psi_0\rangle = 0$. Expressing $|\Psi_0\rangle$ in its Schmidt basis we have that

$$|\Psi_0\rangle = \sum_{j=0}^{1} \sqrt{\lambda_{0j}}|u_{Aj}\rangle \otimes |u_{Bj}\rangle, \tag{19}$$

where $\{u_{A0}, u_{A1}\}$ and $\{u_{B0}, u_{B1}\}$ are orthonormal bases for Alice's and Bob's spaces, respectively, and λ_{0j} for $j = 0, 1$ are the eigenvalues of the reduced density matrixes. Applying the same reasoning as before, we find that

$$A_1 = |\xi_A\rangle\langle s_A| \qquad B_1 = |\xi_B\rangle\langle s_B|, \tag{20}$$

where $\|\xi_A\| = \|\xi_B\| = 1$. We also have that

$$(\langle s_A| \otimes \langle s_B|)|\Psi_0\rangle = 0. \tag{21}$$

We can gain more information about the vectors $|r_A\rangle$, $|r_B\rangle$, $|s_A\rangle$, and $|s_B\rangle$ by substituting the results of the previous paragraphs into Eqs. (6). This gives us that

$$I_A = |r_A\rangle\langle r_A| + |s_A\rangle\langle s_A| \qquad I_B = |r_B\rangle\langle r_B| + |s_B\rangle\langle s_B|. \tag{22}$$

Now let both sides of the first of these equations act on the vector $|r_A\rangle$,

$$\|r_A\|^2|r_A\rangle + |s_A\rangle\langle s_A|r_A\rangle = |r_A\rangle. \tag{23}$$

The only way this can be true is if either $|r_A\rangle$ is parallel to $|s_A\rangle$ which violates Eq. (22), or if $\langle s_A|r_A\rangle = 0$ and $\|r_A\| = 1$. Therefore, $|s_A\rangle$ is orthogonal to $|r_A\rangle$, and both have norm 1. Henceforth, we shall denote $|s_A\rangle$ by $|r_A^\perp\rangle$, and we have that $\{r_A, r_A^\perp\}$ is an orthonormal basis for Alice's space. Similarly, we find that $\{r_B, r_B^\perp\}$, where $|r_B^\perp\rangle = |s_B\rangle$, is an orthonormal basis for Bob's space.

Now let us examine the failure probabilities. We first express the operator F, defined in Eq. (12) as

$$\begin{aligned} F &= (|r_A\rangle \otimes |r_B^\perp\rangle)(\langle r_A| \otimes \langle r_B^\perp|) + (|r_A^\perp\rangle \otimes |r_B\rangle)(\langle r_A^\perp| \otimes \langle r_B|) \\ &= I - (|r_A\rangle \otimes |r_B\rangle)(\langle r_A| \otimes \langle r_B|) \\ &\quad -(|r_A^\perp\rangle \otimes |r_B^\perp\rangle)(\langle r_A^\perp| \otimes \langle r_B^\perp|). \end{aligned} \tag{24}$$

We first note that if Eqs. (18) and (21) are satisfied, then the condition in Eq. (11) is also satisfied. The failure probability if Charlie sends the state $|\Psi_0\rangle$ is $\langle\Psi_0|F|\Psi_0\rangle$, and if he sends the state $|\Psi_1\rangle$, it is $\langle\Psi_1|F|\Psi_1\rangle$. These probabilities can be expressed as

$$\begin{aligned} \langle\Psi_0|F|\Psi_0\rangle &= 1 - |(\langle r_A| \otimes \langle r_B|)|\Psi_0\rangle|^2 \\ \langle\Psi_1|F|\Psi_1\rangle &= 1 - |(\langle r_A^\perp| \otimes \langle r_B^\perp|)|\Psi_1\rangle|^2. \end{aligned} \tag{25}$$

If each of the states is equally likely, then the total failure probability, p_f, is given by

$$p_f = \frac{1}{2}(\langle\Psi_0|F|\Psi_0\rangle + \langle\Psi_1|F|\Psi_1\rangle). \tag{26}$$

We want to minimize this overall failure probability.

Note that the failure probabilities are unaffected by the choice of the vectors $|\xi_A\rangle, |\xi_B\rangle, |\eta_A\rangle$, and $|\eta_B\rangle$. If we make the choices

$$\begin{aligned} |\xi_A\rangle = |r_A^\perp\rangle \quad |\xi_B\rangle = |r_B^\perp\rangle \\ |\eta_A\rangle = |r_A\rangle \quad |\eta_B\rangle = |r_B\rangle, \end{aligned} \tag{27}$$

then the operators A_j and B_j, where $j = 1, 2$, are projections and the generalized meausrement becomes a von Neumann measurement.

Let us summarize our remaining problem. We want to find a basis for Alice's space, $\{|r_A\rangle, |r_A^\perp\rangle\}$, and one for Bob's space, $\{|r_B\rangle, |r_B^\perp\rangle\}$, that satisfy the conditions

$$\begin{aligned}(\langle r_A^\perp| \otimes \langle r_B^\perp|)|\Psi_0\rangle &= 0 \\ (\langle r_A| \otimes \langle r_B|)|\Psi_1\rangle &= 0.\end{aligned} \tag{28}$$

We can reduce these conditions to the solution of several simple equations. First, expanding $|r_A^\perp\rangle$ and $|r_B^\perp\rangle$ in terms of $|u_{Aj}\rangle$ and $|u_{Bj}\rangle$, respectively, we have

$$|r_A^\perp\rangle = \sum_{j=0}^{1} e_j^* |u_{Aj}\rangle \qquad |r_B^\perp\rangle = \sum_{j=0}^{1} f_j^* |u_{Bj}\rangle. \tag{29}$$

The equations in the previous paragraph become

$$\sum_{j=0}^{1} \sqrt{\lambda_{0j}} e_j f_j = 0 \qquad \sum_{j=0}^{1} \sqrt{\lambda_{1j}} c_j d_j = 0, \tag{30}$$

while the conditions $\langle r_A^\perp | r_A\rangle = 0$ and $\langle r_B^\perp | r_B\rangle = 0$ become

$$\begin{aligned}\sum_{j_1, j_2=0}^{1} c_{j_1} e_{j_2}^* \langle v_{Aj_1} | u_{Aj_2}\rangle = 0 \\ \sum_{j_1, j_2=0}^{1} d_{j_1} f_{j_2}^* \langle v_{Bj_1} | u_{Bj_2}\rangle = 0.\end{aligned} \tag{31}$$

Now define the ratios

$$\begin{aligned}z_1 = \frac{c_1^*}{c_0^*} \quad z_2 = \frac{d_1^*}{d_0^*} \\ z_3 = \frac{e_1^*}{e_0^*} \quad z_1 = \frac{f_1^*}{f_0^*}.\end{aligned} \tag{32}$$

If we now divide Eqs. (30) and (31) by the appropriate product of expansion coefficients, e.g. the first of Eqs. (30) is divided by $e_0 f_0$ and the first of Eqs. (31) is divided by $c_0 e_0^*$, we find

$$\begin{aligned}\sqrt{\lambda_{00}} + \sqrt{\lambda_{01}} z_3 z_4 = 0 \\ \sqrt{\lambda_{10}} + \sqrt{\lambda_{11}} z_1 z_2 = 0 \\ \langle v_{A0} | u_{A0}\rangle + \langle v_{A0} | u_{A1}\rangle z_3 \\ + \langle v_{A1} | u_{A0}\rangle z_1^* + \langle v_{A1} | u_{A1}\rangle z_1^* z_3 = 0 \\ \langle v_{B0} | u_{B0}\rangle + \langle v_{B0} | u_{B1}\rangle z_4 \\ + \langle v_{B1} | u_{B0}\rangle z_2^* + \langle v_{B1} | u_{B1}\rangle z_2^* z_4 = 0.\end{aligned} \tag{33}$$

Given two specific states $|\Psi_0\rangle$ and $|\Psi_1\rangle$, these equations can be solved to find the vectors $|r_A\rangle$, $|r_A^\perp\rangle$, $|r_B\rangle$, and $|r_B^\perp\rangle$.

Let us now consider two examples. In the first we shall suppose that $|\Psi_0\rangle$ and $|\Psi_1\rangle$ have the same Schmidt bases while in the second the Schmidt bases of the two states will be different.

We begin by assuming that our two states are given by

$$\begin{aligned} |\Psi_0\rangle &= \cos\theta_0|00\rangle + \sin\theta_0|11\rangle \\ |\Psi_1\rangle &= \cos\theta_1|00\rangle + \sin\theta_1|11\rangle, \end{aligned} \tag{34}$$

where θ_0 and θ_1 are both between 0 and $\pi/2$. Solving Eqs. (33) for these states, we first find the condition $\tan\theta_0 \tan\theta_1 = 1$, which implies that $\theta_1 = (\pi/2) - \theta_0$. We also find explicit expressions for the vectors

$$\begin{aligned} |r_A\rangle &= c_0^*(|0\rangle + z_1|1\rangle) \\ |r_B\rangle &= d_0^*\left(|0\rangle - \frac{\cot\theta_1}{z_1}|1\rangle\right) \\ |r_A^\perp\rangle &= e_0^*\left(|0\rangle - \frac{1}{z_1}|1\rangle\right) \\ |r_B^\perp\rangle &= f_0^*(|0\rangle + \cot\theta_0 z_1^*|1\rangle), \end{aligned} \tag{35}$$

where the normalization constants are given by

$$\begin{aligned} |c_0|^2 &= \frac{1}{1+|z_1|^2} \\ |d_0|^2 &= \frac{|z_1|^2}{|z_1|^2 + (\cot\theta_1)^2} \\ |e_0|^2 &= \frac{|z_1|^2}{1+|z_1|^2} \\ |f_0|^2 &= \frac{1}{1+(\cot\theta_1)^2|z_1|^2}. \end{aligned} \tag{36}$$

The quantity z_1 is at the moment undetermined, but it will be fixed by requiring the failure probability to be a minimum. This probability is now given by

$$p_f = 1 - \frac{|z_1|^2}{2+2|z_1|^2}\ \frac{(1-(\cot\theta_1)^2)^2}{1+(|z_1|\cot\theta_1)^2}\ \frac{(1-(\tan\theta_1)^2)^2}{1+(|z_1|\tan\theta_1)^2}, \tag{37}$$

where the condition $\theta_0 = (\pi/2) - \theta_1$ has been used to eliminate θ_0. Setting the derivative of p_f with respect to $|z_1|^2$ equal to zero, we find an equation that has only one positive solution, $|z_1|^2 = \cot\theta_1$. Substituting this value into Eq. (37), we find

$$p_f = \sin(2\theta_1) \tag{38}$$

This failure probability should be compared to that when a single joint measurement can be performed on both qubits of the two-qubit states. In that

case, if each of the states is equally likely, then the probability of failing to distinguishing the states is given by the IDP limit

$$p_{fidp} = |\langle\Psi_0|\Psi_1\rangle| = \sin(2\theta_1). \tag{39}$$

Note that this expression is identical to that given in the previous paragraph. Therefore, in this example we can conclude that the failure probability that is achieved by measuring the qubits separately is the same as that when the qubits are measured together.

Now let us see what happens if the states have different Schmidt bases. We shall keep $|\Psi_0\rangle$ as before, but choose $|\Psi_1\rangle$ differently,

$$\begin{aligned} |\Psi_0\rangle &= \cos\theta_0|00\rangle + \sin\theta_0|11\rangle \\ |\Psi_1\rangle &= \cos\theta_1|+x\rangle|+x\rangle + \sin\theta_1|-x\rangle|-x\rangle, \end{aligned} \tag{40}$$

where $|\pm x\rangle = (1/\sqrt{2})(|0\rangle \pm |1\rangle)$. Solving Eqs. (33) for these states, we first find a quadratic equation for z_1

$$(1-\cot\theta_0)z_1^2 - (1-\cot\theta_1)(1+\cot\theta_0)z_1 - (1-\cot\theta_0)\cot\theta_1 = 0. \tag{41}$$

The vectors making up the POVM are given by

$$\begin{aligned} |r_A\rangle &= c_0^*(|+x\rangle + z_1|-x\rangle) \\ |r_B\rangle &= d_0^*\left(|+x\rangle - \frac{\cot\theta_1}{z_1}|-x\rangle\right) \\ |r_A^\perp\rangle &= e_0^*(|0\rangle + z_3|1\rangle) \\ |r_B\rangle &= f_0^*\left(|0\rangle - \frac{\cot\theta_0}{z_3}|1\rangle\right). \end{aligned} \tag{42}$$

The normalization constants are given by

$$\begin{aligned} |c_0|^2 &= \frac{1}{1+|z_1|^2} \qquad |e_0|^2 = \tfrac{1}{1+|z_3|^2} \\ |d_0|^2 &= \frac{|z_1|^2}{|z_1|^2+\cot^2\theta_1} \quad |f_0|^2 = \tfrac{|z_3|^2}{|z_3|^2+\cot^2\theta_0}, \end{aligned} \tag{43}$$

where

$$z_3 = -\frac{1-\cot\theta_0\cot\theta_1 + (1-\cot\theta_0)z_1^*}{1-\cot\theta_1}. \tag{44}$$

The failure probability is given by Eqs. (25) and (26), where

$$\begin{aligned} |(\langle r_A| \otimes \langle r_B|)\Psi_0\rangle|^2 &= \frac{|z_1|^2\sin^2\theta_0}{4(1+|z_1|^2)(|z_1|^2+\cot^2\theta_1)} \\ &\left|(1+\cot\theta_0)(1-\cot\theta_1) + (\cot\theta_0-1)\left(z_1^* - \frac{\cot\theta_1}{z_1^*}\right)\right|^2 \\ |(\langle r_A^\perp| \otimes \langle r_B^\perp|)\Psi_0\rangle|^2 &= \frac{|z_3|^2\sin^2\theta_1}{4(1+|z_3|^2)(|z_1|^3+\cot^2\theta_0)} \\ &\left|(1+\cot\theta_1)(1-\cot\theta_0) + (\cot\theta_1-1)\left(z_3 - \frac{\cot\theta_0}{z_3}\right)\right|^2. \end{aligned} \tag{45}$$

Specializing to the case $\theta_0 = \pi/2$ we find that there are two sets of values for $z_1, \ldots z_4$. One set is obtained from the other simply by reversing the roles of $|r_A\rangle$ and $|r_B\rangle$, and both give the same failure probability, so that we need only consider one of them. Doing so we have that

$$\begin{aligned} z_1 &= \cot\theta_1 & z_2 &= -1 \\ z_3 &= \frac{1+\cot\theta_1}{\cot\theta_1 - 1} & z_4 &= 0. \end{aligned} \tag{46}$$

This gives a value for the failure probability of

$$p_f = 1 - \frac{(1-\cot\theta_1)^2 + (\cos\theta_1\cot\theta_1 - \sin\theta_1)^2}{4(1+\cot^2\theta_1)}. \tag{47}$$

This can be compared to the failure probability when both qubits are measured together, which corresponds to the case considered by Ivanovic, Dieks and Peres

$$p_{fidp} = |\langle\Psi_0|\Psi_1\rangle| = \frac{1}{2}(\sin\theta_1 + \cos\theta_1). \tag{48}$$

These probabilities are plotted as a function of $\theta_1$in Fig. 1, and it can be seen that, as expected, $p_f \geq p_{fidp}$. The probabilities are equal at some isolated points, but, in general, there is a cost, which manifests itself as a higher failure probability, associated with determining the state by performing independent measurements on the two particles. This example differs from our previous one in that here there is a difference between p_f and p_{fidp}, whereas there is none when the two states we are trying to distinguish share the same Schmidt basis.

3.2 One failure state

Let us now consider the case in which only one of the four measurement alternatives corresponds to failure. In particular, suppose that $\{0,0\}$ and $\{1,1\}$ correspond to $|\Psi_0\rangle$, $\{1,0\}$ corresponds to $|\Psi_1\rangle$, and $\{0,1\}$ corresponds to failure. We now have the conditions for our POVM operators

$$\begin{aligned} A_0B_0|\Psi_1\rangle &= 0 & A_1B_1|\Psi_1\rangle &= 0 \\ A_1B_0|\Psi_0\rangle &= 0. \end{aligned} \tag{49}$$

Using the same methods as before, we find that

$$\begin{aligned} A_0 &= |r_A\rangle\langle r_A| & A_1 &= |r_A^\perp\rangle\langle r_A^\perp| \\ B_0 &= |r_B\rangle\langle r_B| & B_1 &= |r_B^\perp\rangle\langle r_B^\perp|. \end{aligned} \tag{50}$$

Where we previously had two conditions on the orthonormal bases $\{|r_A\rangle, |r_A^\perp\rangle\}$ and $\{|r_B\rangle, |r_B^\perp\rangle\}$, we now have three

$$\begin{aligned} (\langle r_A| \otimes \langle r_B|)\Psi_1\rangle &= 0 & (\langle r_A^\perp| \otimes \langle r_B^\perp|)\Psi_1\rangle &= 0 \\ (\langle r_A^\perp| \otimes \langle r_B|)\Psi_0\rangle &= 0. \end{aligned} \tag{51}$$

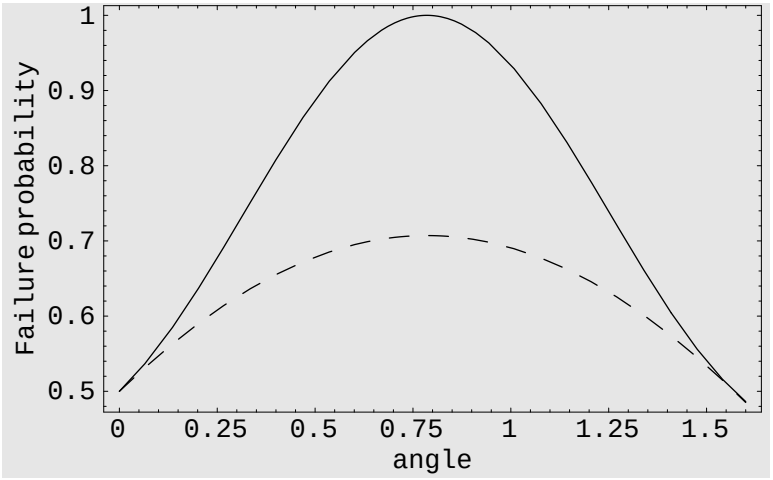

Figure 1: Failure probabilities plotted as a function of the angle θ_1. The solid curve is p_f and the dotted is p_{fidp}. In this case the restriction on classical communication causes an increase in the failure probability.

Let us now consider an example. Let us assume that the states we are trying to distinguish are given by Eq. (34), that is they have the same Schmidt basis. Employing the same methods and notation as before, we find first that $\theta_1 = -\pi/4$, and that

$$\begin{aligned} z_1 &= -z_4^* = \sqrt{\tan\theta_0} \\ z_2 &= -z_3^* = \sqrt{\cot\theta_0} \end{aligned} \tag{52}$$

The failure operator, F is now

$$F = A_0^\dagger A_0 B_1^\dagger B_1 = |r_A\rangle\langle r_A| \otimes |r_B^\perp\rangle\langle r_B^\perp|, \tag{53}$$

where

$$\begin{aligned} |r_A\rangle &= \left(\frac{1}{1+\tan\theta_0}\right)^{1/2} (|0\rangle + \sqrt{\tan\theta_0}|1\rangle) \\ |r_B^\perp\rangle &= \left(\frac{1}{1+\tan\theta_0}\right)^{1/2} (|0\rangle - \sqrt{\tan\theta_0}|1\rangle). \end{aligned} \tag{54}$$

If both states are equally probable, then the failure probability for this procedure is given by

$$\begin{aligned} p_f &= \frac{1}{2}(\langle\Psi_0|F|\Psi_0\rangle + \langle\Psi_1|F|\Psi_1\rangle) \\ &= \frac{1}{2}(\cos\theta_0 - \sin\theta_0)^2 + \frac{1}{4}. \end{aligned} \tag{55}$$

This probability and p_{fidp} are plotted as a function of as a function of θ_0 (θ_1 has been set equal to $-\pi/4$) in Fig. 2.

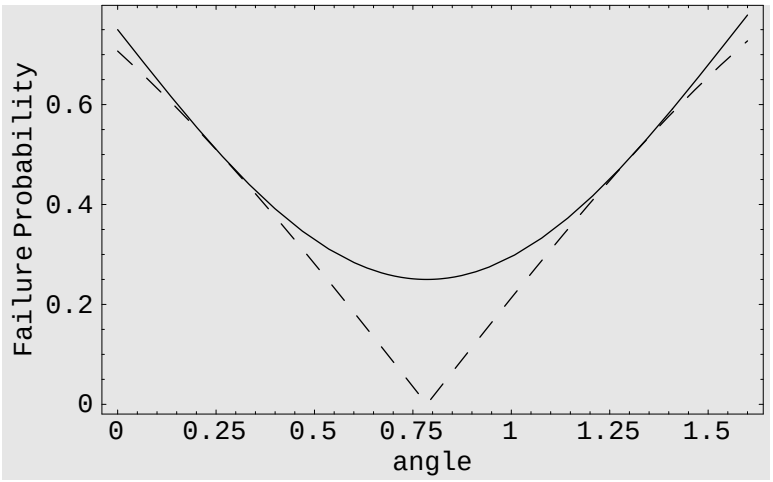

Figure 2: Failure probabilities plotted as a function of the angle θ_0 for the case of one failure state. The solid curve is p_f and the dotted one is p_{fidp}.

4 Secret sharing

There have been a number of theoretical proposals for quantum secret sharing, and one experimental demonstration. The proposals fall into two categories. In the first, quantum mechanics is used to securely distribute a classical shared key. One of these protocols is based on the use of GHZ states [8] and another makes use of pairs of Bell states in different bases [9]. An experiment based on the GHZ state protocol was carried out by Tittel, Zbinden and Gisin [10]. The second category consists of protocols in which the secret information that is split among several parties is quantum information [11]. The procedure we are considering here is of the first type.

Let us suppose that a third party, Charlie, sends one of two states to Alice and Bob, one qubit to Alice and one to Bob,

$$\begin{aligned} |\Psi_0\rangle &= \sin\theta|00\rangle + \cos\theta|11\rangle \\ |\Psi_1\rangle &= \cos\theta|00\rangle + \sin\theta|11\rangle. \end{aligned} \tag{56}$$

The procedure we are discussing here is based on the first example in the preceding section. Initially we shall suppose that Alice measures her state in the basis given by

$$\begin{aligned} |r_A\rangle &= \frac{1}{(1+\cot\theta)^{1/2}}(|0\rangle + \sqrt{\cot\theta}|1\rangle) \\ |r_A^{\perp}\rangle &= \frac{1}{(1+\tan\theta)^{1/2}}(|0\rangle - \sqrt{\tan\theta}|1\rangle), \end{aligned} \tag{57}$$

and that Bob measures his particle in the basis

$$\begin{aligned} |r_B\rangle &= \frac{1}{(1+\cot\theta)^{1/2}}(|0\rangle - \sqrt{\cot\theta}|1\rangle) \\ |r_B^{\perp}\rangle &= \frac{1}{(1+\tan\theta)^{1/2}}(|0\rangle + \sqrt{\tan\theta}|1\rangle). \end{aligned} \tag{58}$$

By comparing their measurement results, Alice and Bob can determine what state Charlie sent, or that the procedure has failed. Individually, however, they will not be able to make this determination. Hence, Alice and Bob together will share a key with Charlie, individually they will not.

Let us now examine the security of this scheme with regard to eavesdropping, and we will quickly see that we have to modify the simple procedure in the previous paragraph. The reason is that an eavesdropper, Eve, has a perfect cheating strategy. Eve simply captures the particles, and performs the same measurement on them that Alice and Bob would perform. She then sends particles to Alice and Bob consistent with her measurement results. For example, if she finds $|r_A\rangle$ and $|r_B\rangle$, she knows the state is $|\Psi_0\rangle$, and she sends a particle in $|r_A\rangle$ to Alice and a particle in $|r_B\rangle$ to Bob. Using this approach, Eve will know the key and Alice, Bob, and Charlie will not be aware of her presence.

This strategy of Eve's can be eliminated if Alice and Bob sometimes measure in the $\{0,1\}$ basis. Each of them chooses randomly, with some predetermined probability, in which basis to measure. When they compare their results, they look at the instances in which they both measured in the $\{0,1\}$ basis, to see if their results were ever different. If they were, they can conclude that an eavesdropper was present. This defeats the attack proposed for Eve in the previous paragraph, because while the states $|\Psi_0\rangle$ and $|\Psi_1\rangle$ have no components along the vectors $|01\rangle$ and $|10\rangle$, states such as $|r_A\rangle|r_B\rangle$ do. That means that in order to avoid detection, Eve must send states lying in the subspace spanned by $|00\rangle$ and $|11\rangle$, which also means that she will not be able to control the results that Alice and Bob get. This leads to her detection. When she measures the state she receives from Charlie and fails, then she has to guess which state to send on to Alice and Bob. Sometimes she will guess incorrectly, and if Alice, Bob and Charlie publicly compare some fraction of their data, they will notice discrepecies, e.g. Charlie will have sent $|\Psi_0\rangle$, but Alice and Bob will have detected $|\Psi_1\rangle$. These discrepencies would not exist if Eve were not present, and their presence gives her away.

Next, let us see whether this procedure protects against cheating. Suppose that Bob is able to capture both qubits sent by Charlie. He first chooses a basis. If it is $\{0,1\}$, he sends a particle to Alice in one of these two states, and throws out the two-qubit state from Charlie (because of his basis choice the results from this state will not contribute to the key). When it comes time to compare results with Alice, if Alice measured the particle Bob sent in the other basis, the results are thrown out, and if she also measured in the $\{0,1\}$ basis, Bob simply announces the result corresponding to the particle he sent her. If Bob chose to measure in the $\{r_A, r_A^{\perp}\}$ and $\{r_B, r_B^{\perp}\}$ bases, then, if he finds $|\Psi_0\rangle$ he send

Alice $|r_A\rangle$, if $|\Psi_1\rangle$, he sends $|r_A^\perp\rangle$, and if he fails he sends either $|r_A\rangle$ or $|r_A^\perp\rangle$. If this is one of the results that is publicly compared, then if Bob's measurement succeeded, he announces the same state as the one he sent to Alice, and if it failed, the opposite state. Using this method, he knows the key bits, and Alice and Charlie do not know that he knows.

It is possible to fix this somewhat if instead of sending the particles to Alice and Bob simultaneously, Charlie first sends one particle to one party, who measures it and tells Charlie over a public channel that he or she has received and measured the particle. Charlie alternates sending the first particle to Alice and Bob. Now, supposing as before that Bob is the cheater, let us see what happens when the particle is sent to Alice first. Bob grabs the particle that has been sent to Alice, but then he must send her a substitute. If he sends her a particle in one of the states $|0\rangle$ or $|1\rangle$, there is no problem, but he cannot do this all of the time, because then no key bits would be generated. If he sends her a particle in either $|r_A\rangle$ or $|r_A^\perp\rangle$, he can run into a difficulty. Suppose he sent her $|r_A\rangle$, and when he receives the second particle from Charlie, he finds that the state Charlie sent was $|\Psi_1\rangle$, which should correspond to Alice measuring $|r_A^\perp\rangle$. If he is to avoid creating a detectable error, he must claim, if this is one of the bits which is publicly revealed, that he measured $|r_B^\perp\rangle$, which corresponds to failure to distinguish. This, however, means that there will be more cases of failure to distinguish than there should be, and Alice and Charlie would be alerted to the fact that the security of the key is questionable.

Instead of sending Alice a single paricle in a specific state, Bob can send Alice one of two particles in a singlet state. This, however, does not help him. From the particle remaining in his posession, he cannot determine which measurement Alice made, because his particle could be in one of four possible states, and these cannot all be orthogonal.

In summary, the procedure outlined here provides protection against eavesdropping, and some protection against cheating. The presence of an eavesdropper leads to errors (misidentification of states) while the presence of a cheater leads to an increased failure rate.

5 Conclusion

We have examined the problem of distinguishing between two two-qubit states without error by using local measurements and either no or limited classical communication. In the first case we found that only one of the two states can be identified, the other generates a failure indication. In the second case, for some pairs of states it is possible to identify the states with the lowest possible failure probability (the IDP limit), and for others the failure probability with limited classical communication is higher than the optimal value. Finally, we proposed a secret sharing scheme based on the procedure using limited classical communication.

Natural generalizations of this work are to higher dimensions, to more than two states, and to states with more than two particles. Many of our results

rely explicitly on the fact that we are considering qubits, and the extension to qudits is not straightforward. For example, we found that with bipartite qubit states it is not possible to distinguish two non-orthogonal states without using classical communication. We could tell if we had one of the two, but if the other state was sent our procedure would always fail. However, if we consider qutrits, whose basis states are $|0\rangle$, $|1\rangle$, and $|2\rangle$, then the two bipartite states

$$\begin{aligned} |\Psi_0\rangle &= \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |2\rangle|2\rangle) \\ |\Psi_1\rangle &= \frac{1}{\sqrt{2}}(|1\rangle|1\rangle + |2\rangle|2\rangle), \end{aligned} \tag{59}$$

which are not orthogonal, can be distinguished without classical communication. Measuring in the basis $\{|0\rangle, |1\rangle, |2\rangle\}$, Alice and Bob will always obtain the same result, and if they obtain $|0\rangle$, they know that $|\Psi_0\rangle$ was sent, if they obtain $|1\rangle$, then $|\Psi_1\rangle$ was sent, and if they obtain $|2\rangle$, then they have failed. The extension to more than two states also introduces new elements. For example, Ghosh, et al. have shown that it is not possible to deterministically distinguish either three or four orthogonal two-qubit states using only local operations and classical communication [12]. This suggests that there is much still to be learned about distinguishing multipartite states using local operations and classical communication.

Acknowledgment

This research was supported by the National Science Foundation under grant number PHY-0139692 and by a PSC-CUNY grant.

Appendix

We now want to show that if no classical communication is permitted, then at most one state can be identified. We begin by using the conditions on the states and POVM operators to derive additional, simpler ones. For example, we have that

$$A_0 B_0 |\Psi_1\rangle = 0 \qquad A_0 B_1 |\Psi_1\rangle = 0. \tag{60}$$

Acting of the first of these with $B_0^\dagger$, the second with $B_1^\dagger$, adding, and making use of Eq. (1), we find that

$$\begin{aligned} 0 &= A_0(I_B - B_f^\dagger B_f)|\Psi_1\rangle \\ &= A_0|\Psi_1\rangle, \end{aligned} \tag{61}$$

where, in going from the first to the second line, we noted that $A_0 B_f |\Psi_1\rangle = 0$. Similarly we find that

$$\begin{aligned} B_0|\Psi_1\rangle = 0 \quad A_1|\Psi_0\rangle = 0 \\ B_1|\Psi_0\rangle = 0. \end{aligned} \tag{62}$$

The next step is to express the states $|\Psi_j\rangle$, where $j = 0, 1$, in terms of their Schmidt bases (see Section III)

$$\begin{aligned} |\Psi_0\rangle &= \sum_{j=0}^{1} \sqrt{\lambda_{0j}} |u_{Aj}\rangle \otimes |u_{Bj}\rangle \\ |\Psi_1\rangle &= \sum_{j=0}^{1} \sqrt{\lambda_{1j}} |v_{Aj}\rangle \otimes |v_{Bj}\rangle. \end{aligned} \tag{63}$$

Application of the two conditions on $|\Psi_1\rangle$ in the previous paragraph imply:

i. If $\lambda_{10} \neq 0$ and $\lambda_{11} \neq 0$, then $A_0|v_{Aj}\rangle = 0$, for $j = 0, 1$, and this implies that $A_0 = 0$. We also have that $B_0 = 0$.

ii. If one of the λ_{1j}'s is zero, and we can assume, without loss of generality, that $\lambda_{11} = 0$, then we have that $A_0|v_{A0}\rangle = B_0|v_{B0}\rangle = 0$.

Similarly, the two conditions on $|\Psi_0\rangle$ give us:

iii. If $\lambda_{00} \neq 0$ and $\lambda_{01} \neq 0$, then $A_1 = B_1 = 0$.

iv. If $\lambda_{01} = 0$, then $A_1|u_{A0}\rangle = B_1|u_{B0}\rangle = 0$.

We now have a number of cases to examine. If conditions (i) and (iii) are true, the only nonzero operators are the failure operators, so that the procedure fails all the time. If conditions (ii) and (iv) are satisfied we have that the POVM operators A_j and B_j must be of the form

$$\begin{aligned} A_0 &= |\xi_A\rangle\langle v_{A1}| \quad B_0 = |\xi_B\rangle\langle v_{B1}| \\ A_1 &= |\eta_A\rangle\langle u_{A1}| \quad B_1 = |\eta_B\rangle\langle u_{B1}|, \end{aligned} \tag{64}$$

where the vectors $|\xi_A\rangle$, $|\xi_B\rangle$, $|\eta_A\rangle$, and $|\eta_B\rangle$ are as yet undetermined.

We now examine the consequences of the conditions $A_0 B_f|\Psi_0\rangle = 0$ and $A_1 B_f|\Psi_1\rangle = 0$, or

$$\begin{aligned} A_0|u_{A0}\rangle \otimes B_f|u_{B0}\rangle &= 0 \\ A_1|v_{A0}\rangle \otimes B_f|v_{B0}\rangle &= 0. \end{aligned} \tag{65}$$

The first of these equations implies that either $\langle v_{A1}|u_{A0}\rangle = 0$, which further implies that, up to a constant of modulus one, $|v_{A0}\rangle = |u_{A0}\rangle$, or that $B_f|u_{B0}\rangle = 0$. If the first alternative is true, then both A_0 and A_1 acting on either vector $|\Psi_j\rangle$ gives zero, and the measurement always fails. If this alternative is to be avoided, then we must have $B_f|u_{B0}\rangle = 0$. However, the second equation tells us that, if the measurement does not always fail, that $B_f|v_{B0}\rangle = 0$. These conditions imply that (assuming that $|u_{B0}\rangle \neq |v_{B0}\rangle$; if this is not true the measurement always fails) $B_f = 0$. We then have that $I_B = B_0^\dagger B_0 + B_1^\dagger B_1$, which can only be true if $|v_{B1}\rangle = |u_{B0}\rangle$ or $|v_{B0}\rangle = |u_{B1}\rangle$, so that $\langle\Psi_0|\Psi_1\rangle = 0$. Sumarizing, we can say that if (ii) and (iv) are satisfied, which implies that $|\Psi_0\rangle$ and $|\Psi_1\rangle$

are product states, then either they are orthogonal, or the measurement always fails.

Finally, let us see what happens if (i) and (iv) are true (the final alternative, (ii) and (iii) being true is equivalent). This implies that $A_0 = B_0 = 0$, so that $|\Psi_0\rangle$ is never detected, and that $|\Psi_0\rangle$ is a product state. Using techniques similar to those in the previous paragraphs, we find that

$$\begin{aligned} A_1 &= |\eta_A\rangle\langle u_{A1}| \quad B_1 = |\eta_B\rangle\langle u_{B1}| \\ A_f &= |\xi_A\rangle\langle u_{A0}| \quad B_f = |\xi_B\rangle\langle u_{B0}|, \end{aligned} \tag{66}$$

where the vectors $|\xi_A\rangle$, $|\xi_B\rangle$, $|\eta_A\rangle$, and $|\eta_B\rangle$ are undetermined unit vectors. The final conditions are given by using the above expressions in the equations $A_1 B_f |\Psi_1\rangle = 0$ and $A_f B_1 |\Psi_1\rangle = 0$ to give

$$\begin{aligned} (\langle u_{A1}| \otimes \langle u_{B0}|)|\Psi_1\rangle &= 0 \\ (\langle u_{A0}| \otimes \langle u_{B1}|)|\Psi_1\rangle &= 0. \end{aligned} \tag{67}$$

An example satisfying these conditions is given in Section II.

References

[1] I. D. Ivanovic, Phys. Lett. A **123**, 257 (1987).

[2] D. Dieks, Phys. Lett. A **126**, 303 (1988).

[3] A. Peres, Phys. Lett. A **128**, 19 (1988).

[4] J. Walgate, A. Short, L. Hardy, and V. Vedral, Phys. Rev. Lett. **85**, 4972 (2000).

[5] S. Virmani, M. F. Sacchi, M. B. Plenio, and D. Markham, quant-ph/0102073.

[6] Yi-Xin Chen and Dong Yang, Phys. REv. A **65**,022320 (2002).

[7] C. H. Bennett, Phys. Rev. Lett. **68**, 3121 (1992).

[8] M. Hillery, V. Bužek, and A. Berthiaume, Phys. Rev. A **59**,1829 (1999).

[9] A. Karlsson, M. Koashi, and N. Imoto, Phys. Rev. A **59**, 162 (1999).

[10] W. Tittel, H. Zbinden, and N. Gisin, Phys. Rev. A **63**, 042301 (2001).

[11] R. Cleve, D. Gottesman, and H. -K. Lo, Phys. Rev. Lett. **83**, 648 (1999).

[12] S. Ghosh, G. Kar, A. Roy, D. Sarkar, A. Sen(De), and U. Sen, Phys. Rev. A **65**, 062307 (2002).